Reflectance imaging at superficial depths in strongly scattering media

Arnold D. Kim

UC Merced Applied Math

Collaborators: Pedro González-Rodríguez, Boaz Ilan, Miguel Moscoso, and Chrysoula Tsogka

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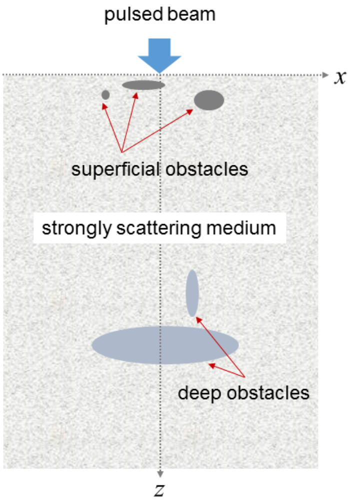
Motivation

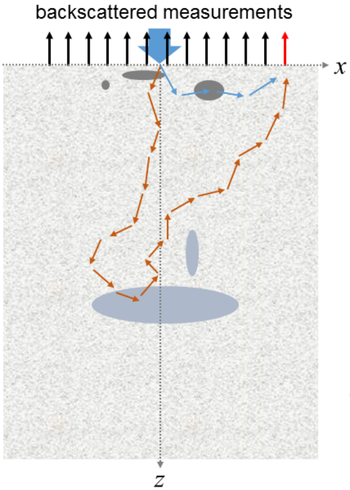
- Imaging in multiple scattering media is important for several different applications, *e.g.* biomedical optics, geophysical remote sensing through clouds, fog, rain, the ocean, etc.
- Strong multiple scattering causes image blurring and makes the inverse problem severely ill-posed.
- There are important problems involving reflectance imaging and spectroscopy at superficial depths, *e.g.* site-specific screening of pre-cancer in epithelial tissues.
- Another way to think of this problem is *near-field imaging in strongly scattering media*.

Imaging problem

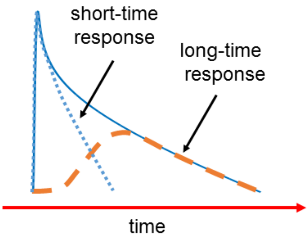


A pulsed beam illuminates the surface of a half space composed of a strongly scattering medium containing superficial and deep obstacles.

Imaging problem



We take time dependent measurements of the light backscattered by this medium on the boundary.



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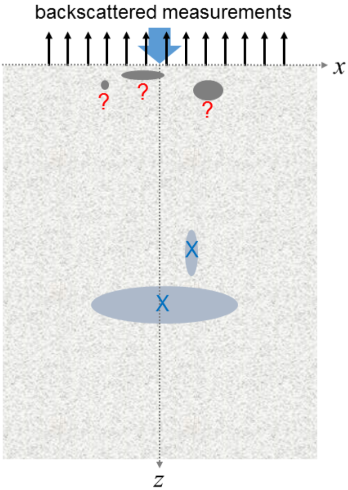
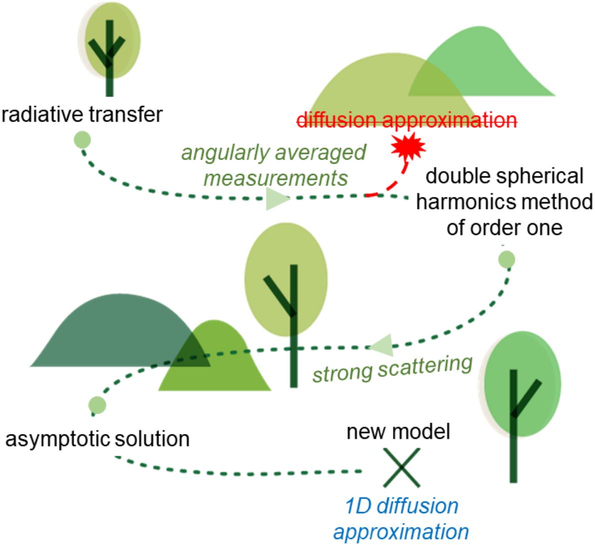


Given these time dependent measurements, we seek to recover *only* the superficial obstacles in the medium.

Challenges

- Majority of backscattered light is diffuse and obscures the obstacles.
- Standard models based on the diffusion approximation are not accurate near sources or boundaries.
- Need a “better” model for this problem.

Modeling roadmap



Radiative transfer theory

- Developed in the early 20th century to describe light scattering by planetary atmospheres.
- It takes into account scattering and absorption by inhomogeneities.
- This theory assumes no phase coherence in its description of power transport (addition of power).
- The specific intensity $I(\mathbf{\Omega}, \mathbf{r}, t)$ quantifies the power flowing in direction $\mathbf{\Omega}$, at position $\mathbf{r}$, and at time t.

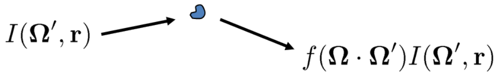
The radiative transfer equation (RTE)

$$c^{-1}\partial_t I + \mathbf{\Omega}\cdot\nabla I + \mu_a I + \mu_s \underbrace{\left[I - \int_{S^2} f(\mathbf{\Omega}\cdot\mathbf{\Omega}') I(\mathbf{\Omega}', \mathbf{r}) \mathrm{d}\mathbf{\Omega}' \right]}_{LI} = 0.$$

- c is the speed of light in the background
- μ_a is the absorption coefficient
- μ_s is the scattering coefficient
- f is the scattering phase function

Scattering phase function

The scattering phase function f gives the fraction of light scattered in direction $\mathbf{\Omega}$ due to light incident in direction $\mathbf{\Omega}'$.



The scattering phase function is normalized according to

$$\int_{S^2} f(\mathbf{\Omega} \cdot \mathbf{\Omega}') \mathrm{d}\mathbf{\Omega}' = 1.$$

We introduce the anisotropy factor, g, defined as

$$\int_{S^2} f(\mathbf{\Omega} \cdot \mathbf{\Omega}') \mathbf{\Omega} \cdot \mathbf{\Omega}' \mathrm{d}\mathbf{\Omega}' = g.$$

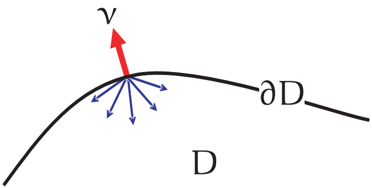
Boundary conditions

To solve

$$c^{-1}\partial_t I + \mathbf{\Omega}\cdot\nabla I + \mu_a I + \mu_s L I = 0$$

in a domain D with boundary ∂D, we prescribe boundary conditions of the form

$$I = I_b \quad \text{on } \Gamma_{\text{in}} = \{(\mathbf{\Omega}, \mathbf{r}, t) \in S^2 \times \partial D \times (0, T], \mathbf{\Omega}\cdot\nu < 0\}.$$



In other words, we must prescribe the light “going into” the medium from the boundary.

Initial-boundary value problem for the RTE

Let $D = \{z > 0\}$ with $\partial D = \{z = 0\}$. Our model for the imaging problem is

$$c^{-1}\partial_t I + \mathbf{\Omega} \cdot \nabla I + \mu_a I + \mu_s L I = 0 \quad \text{in } S^2 \times D \times (0, T],$$

$$I|_{z=0} = \delta(\mathbf{\Omega} - \hat{z}) b(x, y) p(t) \quad \text{on } \Gamma_{\text{in}} = \{S^2 \times \partial D \times (0, T], \mathbf{\Omega} \cdot \hat{z} > 0\}$$

$$I|_{t=0} = 0 \quad \text{on } S^2 \times D$$

$$I \to 0 \quad \text{as } z \to \infty.$$

- The boundary condition prescribes a pulsed beam incident normally on the boundary plane, $z = 0$.
- There is no other source of light in the problem.
- Backscattered light corresponds to $I|_{z=0}$ for directions satisfying $\mathbf{\Omega} \cdot \hat{z} < 0$.

Measurements

Measurements of backscattered light take the form:

$$R(x,y,t) = -\int_{\mathrm{NA}} I(\boldsymbol{\Omega},x,y,0,t)\boldsymbol{\Omega}\cdot\hat{z}\,\mathrm{d}\boldsymbol{\Omega},$$

with $\mathrm{NA} \subset \{\boldsymbol{\Omega}\cdot\hat{z} < 0\}$ denoting the numerical aperture of the detector.

Suppose we measure two or more NAs so that we can recover

$$I_0^- = \frac{1}{\sqrt{2\pi}}\int_{\boldsymbol{\Omega}\cdot\hat{z}<0} I(\boldsymbol{\Omega},x,y,0,t)\,\mathrm{d}\boldsymbol{\Omega}$$

and

$$I_1^- = -\sqrt{\frac{3}{2\pi}}\int_{\boldsymbol{\Omega}\cdot\hat{z}<0} I(\boldsymbol{\Omega},x,y,0,t)\boldsymbol{\Omega}\cdot\hat{z}\,\mathrm{d}\boldsymbol{\Omega}.$$

We take I_0^- and I_1^- as our measurements.

RTE with angularly averaged measurements

- The angularly averaged measurements remove useful direction information from backscattered light, *e.g.* direction dependence of the source.
- Solving the full RTE is unnecessarily complicated for this problem if we only measure I_0^- and I_1^-.
- Using only these measurements makes the inverse problem for the RTE underdetermined.
- The key is to develop the simplest model for measurements that accurately captures the key features of angularly averaged measurements of backscattered light.

Diffusion approximation of the RTE

The diffusion approximation assumes that scattering is so strong that

$$I(\mathbf{\Omega}, \mathbf{r}, t) \sim U(\mathbf{r}, t) + \mathbf{\Omega} \cdot [\kappa \nabla U(\mathbf{r}, t)],$$

where U satisfies

$$c^{-1} U_t + \mu_a U - \nabla \cdot (\kappa \nabla U) = 0.$$

Because $U + \mathbf{\Omega} \cdot (\kappa \nabla U)$ is unable to satisfy boundary condition,

$$I|_{z=0} = \delta(\mathbf{\Omega} - \hat{z}) p(t) \quad \text{on } \Gamma_{\text{in}},$$

we must introduce an approximate boundary condition.

This approximate boundary condition causes errors that make the diffusion approximation unsuitable near sources and boundaries*.

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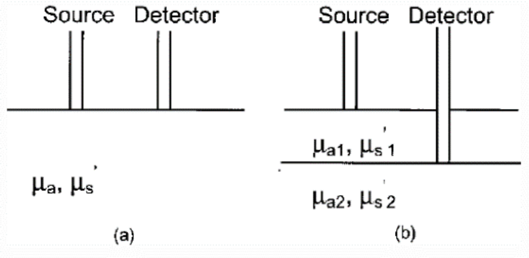
*Rohde and Kim (2012)

Making the diffusion approximation work

S.-H. Tseng and A. J. Durkin† developed a method to circumvent the problem with using the diffusion approximation.



This innovation was used for a fiber-based probe for diffuse optical spectroscopy in epithelial tissues.

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†S.-H. Tseng *et al* (2005) [with permission]

Correcting the diffusion approximation

- The diffusion approximation is a significant simplification over the RTE.
- It is *not wrong* for this problem. Light that penetrates deep into the strongly scattering medium is diffusive.
- It is just not sophisticated enough.
- We could consider the inverse problem for the RTE, but that will require more work than is worthwhile.
- How can we construct a better model?

Double-spherical harmonics method

Since

- Boundary condition prescribes light on $\{\mathbf{\Omega} \cdot \hat{z} > 0\}$,
- Measurements are integrals over $\{\mathbf{\Omega} \cdot \hat{z} < 0\}$,

we write

$$I^{\pm}(\mathbf{\Omega}, \mathbf{r}, t) = I(\pm\mathbf{\Omega}, \mathbf{r}, t), \quad \mathbf{\Omega} \in S_+^2 = \{\mathbf{\Omega} \cdot \hat{z} > 0\},$$

and seek both $I^{\pm}$ as expansions in spherical harmonics, $\{\tilde{Y}_{nm}\}$, mapped to the hemisphere, S_+^2:

$$I^{\pm} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \tilde{Y}_{nm}(\mathbf{\Omega}) I_{nm}(\mathbf{r}, t), \quad \mathbf{\Omega} \in S_+^2.$$

By truncating these expansions at $n = N$, we obtain the double-spherical harmonics approximation of order N (DP_N).

The DP_1 approximation

The simplest approximation is DP_1:

$$I^{\pm} = \sum_{n=0}^{3} \Phi_n(\mu, \varphi) I_n^{\pm}(\mathbf{r}, t),$$

$$\Phi_0 = 1/\sqrt{2\pi}, \quad \Phi_1 = \sqrt{3/2\pi}(2\mu - 1),$$

$$\Phi_2 = \sqrt{3/2\pi}\sqrt{1-\mu^2}\cos\varphi, \quad \Phi_3 = \sqrt{3/2\pi}\sqrt{1-\mu^2}\sin\varphi.$$

Here, $\mu = \cos\theta$ denote the cosine of the polar angle, and φ denote the azimuthal angle.

Note that $I_0^-|_{z=0}$ and $I_1^-|_{z=0}$ are the measurements.

$\Phi_{2,3}$ used here are a slight modification to those typically used in the DP_1 approximation.

The DP_1 system

Substituting $I^{\pm} = \sum_{n=0}^{3} \Phi_n(\mu,\varphi) I_n^{\pm}(\mathbf{r},t)$, into the RTE and projecting onto the finite dimensional subspace, we obtain[‡]

$$\begin{bmatrix} \mathbf{I}^+ \\ \mathbf{I}^- \end{bmatrix}_t + \begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix} \begin{bmatrix} \mathbf{I}^+ \\ \mathbf{I}^- \end{bmatrix}_z + \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} \mathbf{I}^+ \\ \mathbf{I}^- \end{bmatrix}_x + \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} \mathbf{I}^+ \\ \mathbf{I}^- \end{bmatrix}_y$$
$$+ \mu_a \begin{bmatrix} \mathbf{I}^+ \\ \mathbf{I}^- \end{bmatrix} + \mu_s \left(\begin{bmatrix} \mathbf{I}^+ \\ \mathbf{I}^- \end{bmatrix} - \begin{bmatrix} S_1 & S_2 \\ S_2 & S_1 \end{bmatrix} \begin{bmatrix} \mathbf{I}^+ \\ \mathbf{I}^- \end{bmatrix} \right) = 0,$$

where $\mathbf{I}^{\pm} = (I_0^{\pm}, I_1^{\pm}, I_2^{\pm}, I_3^{\pm})$.

The entries of A, B, and C are known explicitly.

S_1 and S_2 are projections of the scattering phase function onto the finite dimensional subspace. Those are computed numerically.

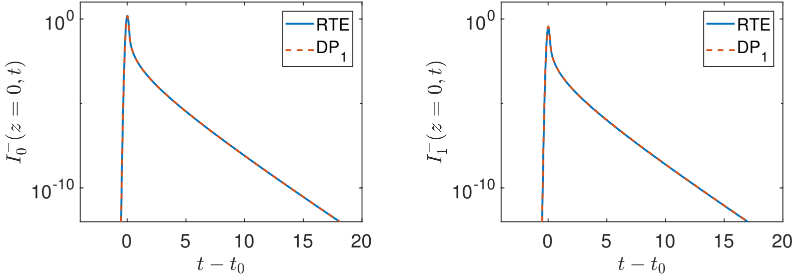
‡Sandoval and Kim (2015)

Solving the DP_1 system

- The DP_1 system is a highly structured, finite dimensional system of forward-backward advection equations.
- It is much simpler problem to solve than the RTE.
- It directly models the measurements.
- Provided it is accurate, its use for imaging problems is novel and interesting.
- Even if it is not accurate, it provides the structure of how information is contained in measurements.

Validating the DP_1 approximation

Numerical results for $\mu_s = 100$, $\mu_a = 0.01$, and $g = 0.8$.



The RTE uses a $\delta(\mathbf{\Omega} - \hat{z})$ source, and the DP_1 approximation uses the projection of this source onto the finite dimensional basis.

DP_1 has errors at short times (not shown here), but it still accurately captures the qualitative behavior of backscattered light.

Strong scattering limit

We introduce $0 < \epsilon \ll 1$ and write $\mu_a = \epsilon\alpha$ and $\mu_s = \epsilon^{-1}\sigma$. We also introduce the slow time $\tau = \epsilon t$ so that the DP_1 system is

$$\epsilon \begin{bmatrix} \mathbf{I}^+ \\ \mathbf{I}^- \end{bmatrix}_\tau + \begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix} \begin{bmatrix} \mathbf{I}^+ \\ \mathbf{I}^- \end{bmatrix}_z + \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} \mathbf{I}^+ \\ \mathbf{I}^- \end{bmatrix}_x + \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} \mathbf{I}^+ \\ \mathbf{I}^- \end{bmatrix}_y$$
$$+ \epsilon\alpha \begin{bmatrix} \mathbf{I}^+ \\ \mathbf{I}^- \end{bmatrix} + \epsilon^{-1}\sigma \left(\begin{bmatrix} \mathbf{I}^+ \\ \mathbf{I}^- \end{bmatrix} - \begin{bmatrix} S_1 & S_2 \\ S_2 & S_1 \end{bmatrix} \begin{bmatrix} \mathbf{I}^+ \\ \mathbf{I}^- \end{bmatrix} \right) = 0,$$

The solution is given as the sum[§]

$$\begin{bmatrix} \mathbf{I}^+ \\ \mathbf{I}^- \end{bmatrix} = \begin{bmatrix} \mathbf{I}^+_{\text{int}} \\ \mathbf{I}^-_{\text{int}} \end{bmatrix} + \begin{bmatrix} \mathbf{I}^+_{\text{bl}} \\ \mathbf{I}^-_{\text{bl}} \end{bmatrix},$$

with $\mathbf{I}^\pm_{\text{int}}$ denoting the interior solution and $\mathbf{I}^\pm_{\text{bl}}$ denoting the boundary layer solution.

[§] Larsen and Keller (1973)

Interior solution

We find that

$$\begin{bmatrix} \mathbf{I}^+_{\text{int}} \\ \mathbf{I}^-_{\text{int}} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_1 \end{bmatrix} (\rho_0 + \epsilon\rho_1)$$
$$- \frac{\epsilon}{\sigma(1-g)} \left\{ \begin{bmatrix} \mathbf{a}_1 \\ -\mathbf{a}_1 \end{bmatrix} \partial_z \rho_0 + \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_1 \end{bmatrix} \partial_x \rho_0 + \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_1 \end{bmatrix} \partial_y \rho_0 \right\} + O(\epsilon^2),$$

with $\hat{e}_1 = (1,0,0,0)$, $\mathbf{a}_1 = A\hat{\mathbf{e}}_1$, $\mathbf{b}_1 = B\hat{\mathbf{e}}_1$, and $\mathbf{c}_1 = C\hat{\mathbf{e}}_1$.

The scalar functions, $\rho_{1,2}$, satisfy

$$\partial_\tau \rho_i + \alpha\rho_i - \nabla \cdot \left(\frac{1}{3\sigma(1-g)} \nabla \rho_i \right) = 0, \quad i = 1,2.$$

We determine that $\rho_0|_{\tau=0} = \rho_1|_{\tau=0} = 0$, but we cannot determine boundary conditions at $z = 0$.

Boundary layer solution

We introduce the stretched variable, $z = \epsilon Z$. The leading order behavior of the boundary layer solution satisfies

$$\begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix} \begin{bmatrix} \mathbf{v}^+ \\ \mathbf{v}^- \end{bmatrix}_Z + \sigma \begin{bmatrix} \mathbb{I} - S_1 & -S_2 \\ -S_2 & \mathbb{I} - S_1 \end{bmatrix} \begin{bmatrix} \mathbf{v}^+ \\ \mathbf{v}^- \end{bmatrix} = 0, \quad \text{in } Z > 0$$

subject to boundary condition

$$\mathbf{v}^+|_{Z=0} = \mathbf{I}^+ b(x,y)p(t) - \hat{\mathbf{e}}_1(\rho_0 + \epsilon\rho_1) + \epsilon \frac{1}{\sigma(1-g)} \mathbf{a}_1 \partial_z \rho_0,$$

and asymptotic matching condition

$$\begin{bmatrix} \mathbf{v}^+ \\ \mathbf{v}^- \end{bmatrix} \to 0, \quad Z \to \infty.$$

This boundary layer solution only depends on x, y, and t parametrically.

Model for measurements

By requiring asymptotic matching, we find that

$$\rho_0|_{z=0} = \alpha_0 b(x,y)p(t), \quad \rho_1|_{z=0} = \alpha_1 \frac{1}{\sigma(1-g)} \partial_z \rho_0|_{z=0}.$$

From these and solving the boundary layer problem, we find that

$$I_0^-|_{z=0} \sim \beta_0 b(x,y)p(t) + \epsilon\beta_1(\kappa\partial_z\rho_0)|_{z=0},$$

and

$$I_1^-|_{z=0} \sim \gamma_0 b(x,y)p(t) + \epsilon\gamma_1(\kappa\partial_z\rho_0)|_{z=0}.$$

Here, $b(x,y)p(t)$ is the known source and $\partial_z\rho_0$ is computed from the diffusion approximation.

Interpreting the model

- In this model, only $(\kappa \partial_z \rho_0)|_{z=0}$ contains any information about the obstacles.

- The boundary layer problem suggests the following.

  - We can directly image in cross-range by scanning.
  - We can isolate the range recovery as a 1D inverse problem for the diffusion approximation.

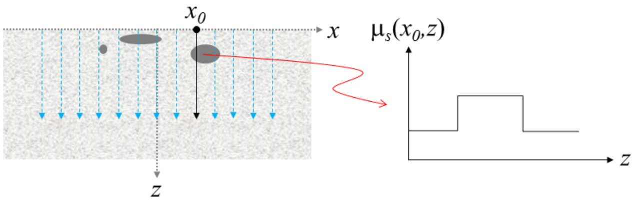
- This reduced model effectively teaches us how to properly apply the diffusion approximation for this imaging problem.

- We obtain the same results for the full RTE in the strong scattering limit[¶].

[¶]Rohde and Kim (2017)

Imaging at superficial depths

The results suggest that we can image at superficial depths by scanning along cross-range, and reconstructing along range.

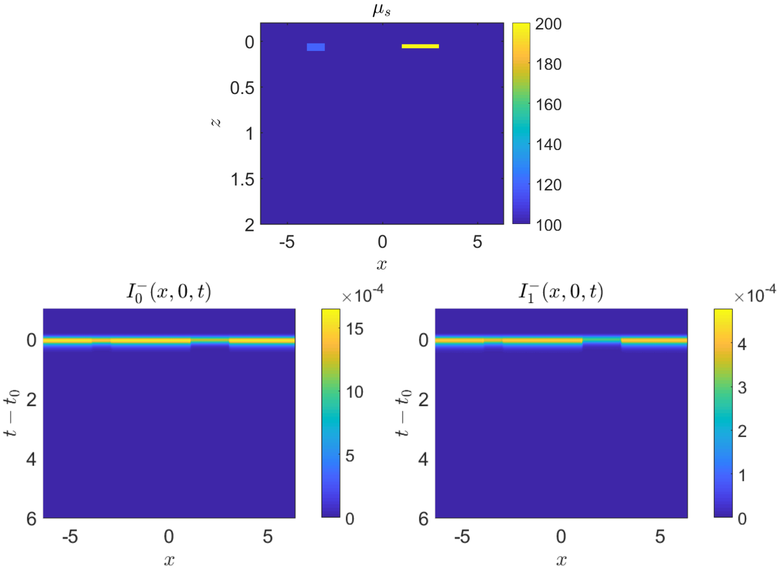


The image reconstruction problem becomes finding κ given measurements $[\kappa(x_0,0)\partial_z\rho]|_{z=0}$ with ρ satisfying

$$\rho_\tau + \alpha\rho - \partial_z[\kappa(x_0,z)\partial_z\rho] = 0,$$

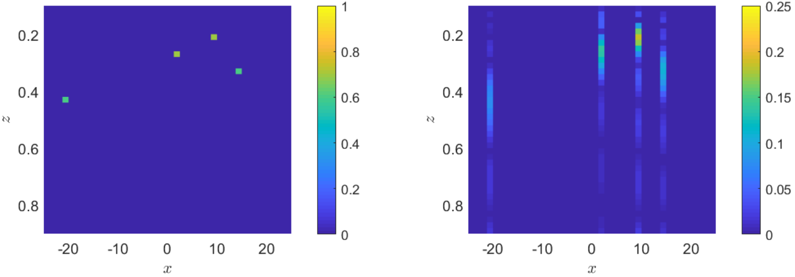
$$\rho|_{\tau=0} = 0, \quad \rho|_{z=0} = \alpha_0 b(x_0)p(t).$$

Direct imaging in cross-range



Preliminary range reconstruction results

We use a simple L_2-based inversion method for the following test case.



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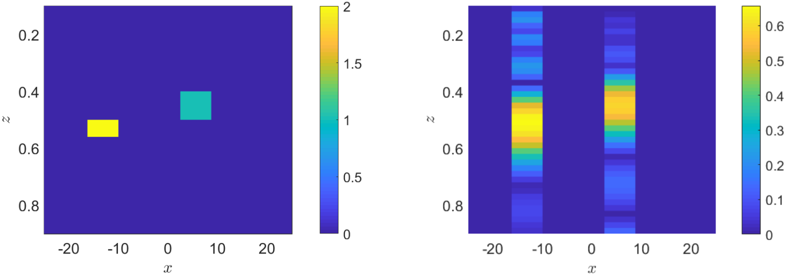

The results are promising, and we are seeking better methods to solve this 1D inverse problem.

Conclusions

- We developed a systematic model for backscattered light measurements using the DP_1 approximation of the RTE.
- The results state that the measurements are linear combinations of the incident pulsed beam, $b(x,y)p(t)$, and the Dirichlet-to-Neumann map of the diffusion equation.
- Boundary layer analysis suggest that imaging at superficial depths only requires direct imaging in cross-range and a 1D reconstruction in range.
- Preliminary results show that this is an efficient method for imaging superficial targets in strongly scattering media.